2. Units, inverses, and homotopes

The defining conditions for a unit or inverse can be phrased in Jordan terms, and since left multiplications respect Jordan products they respect inverses. Indeed, left Moufang division algebras are exactly characterized by the Left Inverse Property $L_{x^{-1}} = L_x^{-1}$. Inverses don't interact so nicely with right multiplications. Homotopes too work well on the left but not the right; a left homotope $A^{(u)}$ remains left Moufang, but a two-sided homotope $A^{(u,v)}$ may not.

Units

An element u is an (ordinary) unit for A if $ux = xu = x$ for all $x \in A$. An element is a Jordan unit if $U_u x = x$, $U_x u = x^2$ for all x.

2.1 Proposition. An element u is an ordinary unit iff it is a Jordan unit.

Proof. Clearly if $ux = xu = x$ for all x we will have $U_u x = u(xu) = ux = x$ and $U_x u = x(ux) = xx = x^2$. Conversely, if u is a Jordan unit then $u^2 = U_u u = u$ so $xu = L_{U(u)x} u = L_u L_x L_u u = u\{xu^2\} = u\{xu\} = U_u x = x$ and therefore $ux = u(xu) = U_u x = x$ too. ∎

2.2 Proposition. A has a unit if either one of

(i) some U_x is surjective

(ii) some L_x, R_x are surjective.

Proof. Since $U_x = L_x R_x$, (ii) implies (i), so we show only that (i) implies the existence of a unit.

There must be a y such that $x = U_x y$, and $y = U_x z$ for some z. Then $L_x L_y L_x = L_{U(x)y} = L_x$ shows $L_x L_y = I$ on $L_x A = A$ (if $U_x = L_x R_x$ is surjective so is L_x), and then $L_y = L_x L_z L_x$ injective implies L_x injective too, so L_x and L_y are inverses. Set $u = yx$. This u is a left unit since $L_u = L_{yx} = L_{x[y(yx)]} = L_{x(y^2x)} = L_x L_y L_y L_x = I$, and a right unit since for any $a = U_x b$ in A we have $au = x\{b(xu)\} = x\{bx\} = a$ by $xu = x(yx) = x$. ∎

2.3 Remark. It is an open question whether the existence of a bijective left multiplication implies the existence of a left unit, i.e.,

$$\text{some } L_x \text{ bijective} \Rightarrow \text{some } L_u = I.$$

The answer is "yes" for algebraic algebras (see Problem Set IV.2.1).

Inverses

Two elements x,y are (ordinary) **inverses** in a unital algebra A if $xy = yx = 1$; they are **Jordan inverses** if $U_x y = x$, $U_x y^2 = 1$.

2.4 Theorem. The following are equivalent in a unital left Moufang algebra A:

(i) x,y are inverses

(ii) x,y are Jordan inverses

(iii) L_x, L_y are inverses.

In this case the inverse $y = x^{-1}$ is unique, and

$$L_{x^{-1}} = R_x U_x^{-1} = L_x^{-1},\ R_{x^{-1}} = L_x U_x^{-1} = L_x R_x^{-1} L_x^{-1},\ U_{x^{-1}} = U_x^{-1}.$$

Proof. (i)$\Rightarrow$(ii): if $xy = yx = 1$ clearly $U_x y = x(yx) = x$ and $U_x y^2 = x\{y^2 x\} = x\{y(yx)\} = xy = 1$.

(ii)$\Rightarrow$(iii): since $x \to L_x$ is a monomorphism $A \to \mathrm{End}(A)$ of quadratic algebras, inverses x,y in A go into inverses L_x, L_y in End(A):

$L_x L_y L_x = L_x, L_x L_y^2 L_x = I$ shows L_x is invertible with inverse L_y in End A.

(iii)$\Rightarrow$(i): $1 = L_x L_y 1 = xy$ and similarly $1 = yx$.

In this case $U_x U_y U_x = U_{U(x)y} = U_x$ and $U_x U_{y^2} U_x = U_{U(x)y^2} = I$ (by the Fundamental Formula (1.5)) imply U_x, U_y are inverses, $U_{x^{-1}} = U_x^{-1}$. We have $L_{x^{-1}} = L_x^{-1}$ by (iii), and $L_x R_x U_x^{-1} = I$ implies $R_x U_x^{-1} = L_x^{-1}$. Cancelling in $L_{x^{-1}} R_{x^{-1}} U_x = U_{x^{-1}} U_x = I$ gives $R_{x^{-1}} = L_x U_x^{-1}$; since L_x and $U_x = L_x R_x$ are invertible, so is R_x, thus $U_x^{-1} = R_x^{-1} L_x^{-1}$ and $R_{x^{-1}} = L_x R_x^{-1} L_x^{-1}$. ∎

Note that if A is not alternative we need not have $R_{x^{-1}} = R_x^{-1}$, although R_x is invertible when x is; indeed R_x^{-1} coincides with its conjugate $R_{x^{-1}} = L_x R_x^{-1} L_x^{-1}$ iff L_x and R_x commute.

The conditions on an element for it to be invertible are given by

2.5 (Inverse Theorem). The following are equivalent for an element x of a unital left Moufang algebra:

(i) x is invertible, $xy = yx = 1$ for some y

(ii) $xz = yx = 1$ for some y,z

(iii) $1 \in \text{Range } L_x \cap \text{Range } R_x$

(iv) $1 \in \text{Range } U_x$

(v) L_x, R_x are invertible

(vi) U_x is invertible.

Proof. Clearly (i)$\Rightarrow$(ii)$\Leftrightarrow$(iii), and (ii)$\Rightarrow$(i) since it implies $xy = x\{y(xz)\} = \{x(yx)\}z = xz = 1$. Clearly (v)$\Rightarrow$(vi)$\Rightarrow$(iv), and we remarked (i)$\Rightarrow$(v) after the previous theorem; (iv)$\Rightarrow$(vi) since if $U_x z = 1$ then $U_x U_z U_x = I$ (via the Fundamental Formula (1.5)) shows U_x has left and right inverse; (vi)$\Rightarrow$(i) since if $U_x y = x$ then $U_x U_y U_x = U_x$ implies $U_x U_y = I$, $U_x y^2 = U_x U_y 1 = 1$, x is Jordan invertible and hence (by the previous theorem) invertible. ∎

Naturally enough, we define a unital left Moufang algebra to be a **division algebra** if all of its nonzero elements are invertible. By (2.5v) this coincides with the general definition of a nonassociative division algebra as one in which all L_x, R_x are invertible.

As in the case of alternative algebras, closeness to being a division algebra is measured by the inner (rather than one-sided) ideals.

2.6 (Division Algebra Criterion) A unital left Moufang algebra is a division algebra iff it has no proper inner ideals.

Proof. A division algebra has no proper inner ideals, since any time an inner ideal B contains an invertible element x it must be the whole algebra: $B \supset U_x A = A$ by invertibility of U_x.

If A is unital and has no proper inner ideals then each principal inner ideal $U_x A$ is 0 or A. If $U_x A = 0$ then Φx is already an inner ideal, consisting entirely of trivial elements; since it doesn't contain 1 it can't be all of A, so it must be 0. Thus $U_x A = 0$ implies $x = 0$, or put another way $x \neq 0$ implies $U_x A \neq 0$ (in which case $U_x A = A$ by impropriety, and x is invertible by the Inverse Theorem (2.5iv)). Consequently all $x \neq 0$ are invertible, and A is a division algebra. ∎

Inverses can be used to characterize left Moufang algebras. A nonassociative division ring is said to have the **left inverse property** if for each element $x \neq 0$ there is an element x^{-1} satisfying

$$x^{-1}(xy) = y \quad (x \neq 0, \text{ all } y) \quad . \tag{2.7}$$

In operator form this means $L_{x^{-1}}$ is the left inverse of L_x,

$$L_{x^{-1}}L_x = I.$$

Then $L_{x^{-1}}$ has a left inverse $L_{(x^{-1})^{-1}}$ and also a right inverse L_x, so $L_{x^{-1}}$ is invertible. But then the above relations says $L_{x^{-1}}$ is the (two-sided) inverse of L_x, so the left inverse property is equivalent to the existence for each x of an x^{-1} with

(2.8) $$L_{x^{-1}} = L_x^{-1} \qquad (x \neq 0).$$

This shows x^{-1} is uniquely determined as the left inverse of x,

$$x'x = 1 \Rightarrow x' = x^{-1}$$

since $x'x = 1 \Rightarrow L_x L_{x'} L_x = L_{x(x'x)} = L_x \Rightarrow L_{x'} = L_x^{-1} = L_{x^{-1}} \Rightarrow x' = x^{-1}$.

2.9 (Left Inverse Property Theorem). A nonassociative algebra has the left inverse property iff it is a left Moufang division algebra.

Proof. Our previous work on inverses has shown every left Moufang division algebra has the left inverse property. Conversely, assume A has the left inverse property. In Problem Set I.4.1 we have already indicated one way to derive the left Moufang formula.

The basic idea is simple: Hua's formula says xyx can be built out of subtraction and inversion, so any map preserving these operations will preserve xyx. But the map $x \to L_x$ preserves subtraction $L_{x-y} = L_x - L_y$ by linearity, and preserves inverses $L_{x^{-1}} = L_x^{-1}$ by the Left Inverse Property (2.8), so it must preserve xyx and we have left Moufangitivity $L_{x(yx)} = L_x(L_yL_x)$.

Thus the key is

2.10 (Hua's Identity) If $x, y, x-y^{-1}$ are invertible elements of an associative algebra then $x-xyx$ is also invertible, with

$$(x-xyx)^{-1} = x^{-1}-(x-y^{-1})^{-1}.$$

Alternately,

$$xyx = x-\{x^{-1}-(x-y^{-1})^{-1}\}^{-1}.$$

Proof. We must show $a = x-xyx$ and $b = x^{-1}-(x-y^{-1})^{-1}$ are inverses. But

$$\begin{aligned} ab &= ax^{-1}-a(x-y^{-1})^{-1} \\ &= \{(1-xy)x\}x^{-1}-\{xy(y^{-1}-x)\}(x-y^{-1})^{-1} \\ &= (1-xy)+xy = 1 \end{aligned}$$

and

$$\begin{aligned} ba &= x^{-1}a-(x-y^{-1})^{-1}a \\ &= x^{-1}\{x(1-yx)\}-(x-y^{-1})^{-1}\{(y^{-1}-x)yx\} \\ &= (1-yx)+yx = 1. \ \blacksquare \end{aligned}$$

Applying Hua's identity to the invertible elements $L_x, L_y, L_x-L_y^{-1}$ of the associative algebra End(A) (where A has left inverse property and $x, y, x-y^{-1} \neq 0$) we get

$$\begin{aligned} L_xL_yL_x &= L_x-\{L_x^{-1}-(L_x-L_y^{-1})^{-1}\}^{-1} \\ &= L_x-[L_{x^{-1}-(x-y^{-1})^{-1}}]^{-1} \qquad \text{(using 2.8))} \\ &= L_{x-[x^{-1}-(x-y^{-1})^{-1}]^{-1}} \qquad \text{(using (2.8)).} \end{aligned}$$

Applying this operator to the element 1 yields

$$x(yx) = x-\{x^{-1}-(x-y^{-1})^{-1}\}^{-1}$$

(so Hua's Identity actually holds in A itself!), and our previous operator equation simplifies to the left Moufang law

$$L_xL_yL_x = L_{x(yx)}.$$

As usual, we have glibly passed over the cases $x = 0$, $y = 0$, $x-y^{-1} = 0$ when the requisite inverses don't exist. But $L_x L_y L_x = L_{x(yx)}$ is trivial if x or y is 0, and if $x = y^{-1}$ then $L_x L_y L_x = L_y^{-1} L_y L_x = L_x = L_{y^{-1}(yx)} = L_{x(yx)}$. ■

Since dually right Moufang is equivalent to right inverse property, and left + right Moufang is equivalent to alternativity, we have the characterization of alternative algebras in terms of inverses which we promised in Section I.4:

2.11 (Inverse Property Theorem) A nonassociative division ring is alternative iff it has the inverse property (i.e., both left and right inverse properties). ■

In the next section we turn to showing that for a division algebra, left Moufang alone already implies alternativity (thus the left inverse property implies alternativity).

Homotopes

Given an element u in a left Moufang algebra A we can endow A with a new multiplicative structure $A^{(u)}$, the **u-homotope** of A, by

(2.12) $$x \cdot_u y = x(uy) \quad .$$

Since $L_x^{(u)} = L_x L_u$ and $U_x^{(u)} y = x \cdot_u (y \cdot_u x) = (L_x L_u)(L_y L_u)x = L_x L_{U(u)y} x = U_x(U_u y)$ we see $A^{(u)}$ is again left Moufang:

$$L^{(u)}_{x^2(u)} = L_{x(ux)} L_u = L_x L_u L_x L_u = L_x^{(u)} L_x^{(u)}$$

$$L^{(u)}_{U^{(u)}(x)y} = L_{U(x)U(u)y} L_u = L_x L_u L_y L_u L_x L_u = L_x^{(u)} L_y^{(u)} L_x^{(u)} \quad .$$

If u is invertible we call $A^{(u)}$ the **u-isotope** of A; here

(2.13) $$1^{(u)} = u^{-1}$$

is the unit for $A^{(u)}$ since $L^{(u)}_{u^{-1}} = L_{u^{-1}}L_u = I$ and $R^{(u)}_{u^{-1}} = R_{uu^{-1}} = I$.

We have transitivity of homotopes

$$(2.14) \qquad \{A^{(u)}\}^{(v)} = A^{(U_u v)}$$

since $L_x^{(u)(v)} = L_x^{(u)}L_v^{(u)} = L_xL_uL_vL_u = L_xL_{U(u)v} = L_x^{(U_u v)}$, and therefore <u>symmetry</u> in the case of isotopes:

$$(2.15) \qquad \{A^{(u)}\}^{(u^{-2})} = A.$$

From this we can conclude that A is alternative iff its isotope $A^{(u)}$ is alternative. For if A is alternative so is any left homotope $A^{(u)}$, and if $A^{(u)}$ is alternative so is its left homotope A.

AIV.2 Exercises

2.1 Establish Proposition 2.2 by Jordan methods as follows. If U_x is surjective and $U_x z = x$, show $U_x y = x$, $U_y x = y$ for $y = U_z x$. Show $U_x U_y = I$. Write $y = U_x w$ to show $U_y U_x = U_x U_y = I$. Conclude $U_u = I$ for $u = U_y x^2$. Show $U_x u = x^2$, then $U_a u = a^2$ for all $a = U_x b$. Deduce that $u = U_y x^2$ is a unit.

2.2. Pick any two parts of the Inverse Theorem 2.5 and show directly one implies the other.

2.3 In an algebra with the left inverse property show directly that x^{-1} is the unique two-sided inverse of x, and $(x^{-1})^{-1} = x$.

2.4 Define the negative powers $x^{-n} = (x^{-1})^n$ for an invertible element x. Show that the rules

$$x^n x^m = x^{n+m}, \; L_{x^n} = L_x^n, \; U_{x^n} = U_x^n, \; x^n = L_x^n 1$$

hold for all (positive or negative) integers. Conclude that the set of all powers x^n spans a commutative associative subalgebra $\Phi[x,x^{-1}]$ of A.

2.5 Show that if A is a left Moufang algebra with no proper inner ideals, then either $A = \Phi z$ is trivial or A is a unital division algebra. In the former case show $\bar{\Phi} = \Phi/\Omega$ $(\Omega = \{\alpha \in \Omega \mid \alpha A = 0\})$ is a field, so A is a 1-dimensional vector space over $\bar{\Phi}$ and thus has no proper subspaces at all.

2.6 The Division Algebra Criterion 2.6 applies to any Jordan algebra. (i) Show that if A is a space with a quadratic map $x \to U_x$ of $A \to \mathrm{End}_\Phi(A)$ satisfying the Fundamental Formula, show any $B = U_x A$ is an inner ideal (in the sense that $U_B A \subset A$), and that if z is trivial $(U_z = 0)$ then $B = \Phi z$ is inner. (ii) If A is unital (with an element $1 \in A$ satisfying

$U_1 = I$), show any element x with $1 \in U_x A$ is invertible (in the sense that U_x is invertible – with inverse U_y when $U_x y = x$).

2.7 Prove directly the Hua Identity

$$x(yx) = x - [x^{-1} - (x - y^{-1})^{-1}]^{-1}$$

when $x, y, x-y^{-1}$ have left inverses in an algebra (not necessarily a division algebra!) satisfying

$$x^{-1}x = 1 \Rightarrow x^{-1}(xy) = y \text{ for all } y.$$

2.8 Give a more elegant proof that the homotope $A^{(u)}$ is again left Moufang along the following lines. Show A is left Moufang iff $x \to L_x$ is a homomorphism $A \xrightarrow{L} End(A)$ of quadratic algebras. Show a homomorphism $A \xrightarrow{F} B$ of quadratic algebras induces a homomorphism $A^{(u)} \xrightarrow{F} B^{(F(u))}$ of quadratic homotopes (where $x^{2(u)} = U_x u$ and $U_x^{(u)} = U_x U_u$). If B is associative show $B^{(v)} \xrightarrow{\cdot v} B$ by $x \to xv$ is a homomorphism of associative (hence also quadratic) algebras. Show that when A is left Moufang the composite $A^{(u)} \xrightarrow{L} End(A)^{(L_u)} \xrightarrow{\cdot L_u} End(A)$ is a homomorphism $x \to L_x^{(u)}$ of quadratic algebras, and deduce from this that $A^{(u)}$ is left Moufang.

2.9 Show x is invertible in an isotope $A^{(u)}$ iff it is invertible in A, with $x^{-1(u)} = U_u^{-1} x^{-1}$; conclude $A^{(u)}$ is a division algebra iff A is. Show a left, right, or two-sided ideal B of A remains such in any homotope $A^{(u)}$; conclude an isotope $A^{(u)}$ has the same one or two-sided ideals as A, in particular is simple or has d.c.c. iff A does.

2.10 Define a right homotope $A^{[v]}$ of a left Moufang algebra A by

$$x_v \cdot y = (xv)y.$$

Find expressions for the operators $L_x^{[v]}, R_x^{[v]}, U_x^{[v]}$ and for the commutator $[x,y]^{[v]}$ and associator $[x,y,z]^{[v]}$. Show $A^{[v]}$ need not even be power associative: $[x,x,x]^{[v]} = [xv,x,v]x = \{[v,x^2,v]-x[v,x,v]\}x$ needn't vanish if A is left Moufang but not alternative. [To see this more clearly, note $[1,1,x]^{[v]}+[1,x,1]^{[v]}+[x,1,1]^{[v]} = 2[v,x,v]$, so choose a unital example of characteristic $\neq 2$]. If v is invertible show $A^{[v]}$ has unit $1^{[v]}$ iff $R_{v^{-1}} = R_v^{-1}$, in which case $1^{[v]} = v^{-1}$; give an example where $A^{[v]}$ is not unital (though v^{-1} is always at least a left unit). Show $\{A^{[v]}\}^{[w]} = A^{[(vw)v]}$ iff $R_vR_wR_v = R_{(vw)v}$. If A is unital show $A^{[v]}$ is left alternative iff $[v,A,v] = 0$.

2.11 Define a (two-sided) homotope $A^{(u,v)}$, find expressions for $L^{(u,v)}$, $R^{(u,v)}$, $U^{(u,v)}$, $[x,y,z]^{(u,v)}$. Find conditions for existence of a unit. Find necessary and sufficient conditions that $A^{(u,v)}$ be again left alternative or left Moufang.